SU(3) Weight Diagrams and Gell-Mann–Okubo Mass Relations for Tetraquark and Pentaquark Multiplets

Varun Rajkumar*

Abstract

This paper examines how SU(3) flavor symmetry can be used to classify tetraquarks and pentaquarks and relate the masses of particles within the same multiplet. Young tableaux are used to determine the possible SU(3) representations, and weight diagrams are constructed to show the allowed states in terms of isospin and hypercharge. The tetraquark, pentaquark, and antipentaquark flavor decompositions are derived, including octet, decuplet, 27-plet, and 35-plet representations. The Gell-Mann–Okubo mass relations are then used to connect the masses of states within these multiplets. For hidden-charm tetraquarks, treating the $Z_c(3900)$ and $Z_{cs}(4000)$ as members of the same flavor octet gives a predicted mass of approximately 4022 MeV for the missing isoscalar partner. For observed hidden-charm pentaquarks, the available experimental states are not yet sufficient to make a similar mass prediction. These results show how flavor symmetry can help organize exotic hadrons and guide searches for new particles.

Introduction

Quantum Chromodynamics (QCD), initially developed in its modern form in 1973, is the theory of how the strong force, which holds quarks together through the exchange of gluons, works [7, 9, 14]. Quarks carry a quantum number known as color, and QCD is based on the symmetry group $SU(3)_{\text{color}}$ [7]. Due to the strong force, quarks cannot be isolated, and they must combine in clusters of two or more to form particles known as hadrons. The most common types of hadrons are baryons and mesons. Baryons, which consist of three quarks and include protons and neutrons, are of the form (qqq), meaning that all their constituents are quarks rather than antiquarks. Mesons, which include pions and kaons, are of the form $(q\bar{q})$, meaning that they consist of one quark and one antiquark.

However, other types of hadrons also exist. In 2003, the LEPS Collaboration found evidence for a hadron called the Θ^+, which, strangely enough, could not be described as a conventional meson or baryon, suggesting the existence of a particle containing more than three quarks [11]. This experimental result was unclear, and later experiments found no convincing evidence for the state [6]. In 2013, the $Z_c(3900)$ was observed by the BESIII experiment in China and the Belle experiment in Japan and was identified as a tetraquark candidate [5, 10]. In 2015, CERN's LHCb experiment reported the observation of pentaquark candidates, which contain five quarks and antiquarks [2]. Thus, these discoveries provided evidence that hadrons beyond conventional mesons and baryons exist, although they are much rarer.

In this paper, I will analyze the properties of pentaquarks using $SU(3)$ flavor weight diagrams and Gell-Mann–Okubo mass relations [8, 13, 12]. Studying the properties of pentaquarks, as well as tetraquarks, is important because it helps us predict which particles may exist and understand how the strong force works within them. For example, by using the properties of flavor $SU(3)$ to classify pentaquarks, we can determine the possible combinations of quark flavors and the multiplets they form [12]. I can also use the Gell-Mann–Okubo mass relations to estimate the

*Homestead High School

masses of pentaquarks within different flavor multiplets [12]. Both calculations help us predict the properties of pentaquarks and identify them in future experiments.

Methods

In order to fully analyze Tetraquarks and Pentaquarks, we must understand how they are structured through the $SU(3)$ group. The $SU(3)$ represents the group of all unitary 3×3 matrices, meaning they have $\det(R) = 1$ and also satisfy $R^{\dagger}R = I$. It turns out that the symmetry of the group strongly describes the Strong force interaction from Quantum Chromodynamics. The 8 gauge bosons that mediate this force are called Gluons, which match the 8 generators of the $SU(3)$ group. In addition, the three fundamental states represent the three colors of quarks: Red, Green, and Blue. However, the SU(3) symmetry used in this paper to classify tetraquarks and pentaquarks is $SU(3)_{\text{flavor}}$. In this approximate symmetry, the three fundamental states correspond to the light quark flavors u, d, and s. Since the u and d quarks have similar masses while the s quark is heavier, $SU(3)_{\text{flavor}}$ is not exact but we can use it to predict what types of Pentaquarks and Tetraquarks exist and how they are structured.

To determine the possible irreducible representations, we can use the Young Tableaux method, where each box represents a quark in the fundamental **3** representation of $SU(3)$. Boxes placed in the same row represent indices that are symmetrized, while boxes placed in the same column represent indices that are antisymmetrized. Tableux containing both rows and columns correspond to mixed representations. For example, combining two quarks gives $\mathbf{3} \otimes \mathbf{3} = \mathbf{6} \oplus \bar{\mathbf{3}}$, where the **6** is symmetric and the $\bar{\mathbf{3}}$ is antisymmetric. In addition, each tableau can be labeled as 2 integers (p, q), where p is the difference between the first row and second row, and q is the difference between the second row and third row. The number of states for this representation can be calculated as $\frac{1}{2}(p+1)(q+1)(p+q+2)$. By combining Tetraquarks and Pentaquarks this way using Young Tableux, then calculating the different combinations, we can use that to figure out the number of states for each, and whether they are symmetric, antisymmetric, or mixed.

One way to represent the effects of these are weight diagrams. Weight diagrams graph the I_3 property of a hadron, which represents its component of isospin it has, to the Y value, which represents its hypercharge. Isospin is calculated through $I_3 = \frac{1}{2}(n_u - n_d)$, while hypercharge is calculated through $B + S$. When making this weight diagrams, it is also observed that particles of different types form similar shapes, such as hexagonal eight-fold pattern, or a triangle or diamond, depending on the case. These patterns govern the types of Pentaquarks and Tetraquarks that exist.

In addition to classifying the flavor multiplets, the masses within each multiplet can be related using the Gell-Mann–Okubo (GMO) mass formula. Flavor $SU(3)$ would imply equal masses if the u, d, and s quarks had identical masses. Since the strange quark is heavier, the symmetry is only approximate. To first order in $SU(3)$ breaking, the mass of a state within a multiplet can be written as

$$M(Y, I) = M_0 + \alpha Y + \beta \left[I(I+1) - \frac{Y^2}{4} \right],$$

where Y is hypercharge, I is isospin, and M_0, α, and β are constants for the multiplet. Thus, once several members of a tetraquark or pentaquark multiplet are identified, their masses can be used to constrain the remaining states.

Results

Tetraquark multiplets: $\mathbf{3} \otimes \mathbf{3} \otimes \bar{\mathbf{3}} \otimes \bar{\mathbf{3}}$

A tetraquark carries the flavour content $qq\bar{q}\bar{q}$, so the relevant product is $\mathbf{3}^{\otimes 2} \otimes \bar{\mathbf{3}}^{\otimes 2}$. Two quarks give the symmetric sextet and the antisymmetric anti-triplet,

$$\ydiagram{1} \otimes \ydiagram{1} = \ydiagram{2} \oplus \ydiagram{1,1}, \qquad \mathbf{3} \otimes \mathbf{3} = (2,0) \oplus (0,1) = \mathbf{6} \oplus \bar{\mathbf{3}}.$$

The conjugate product follows by transposition of the diagrams,

$$\ydiagram{1,1} \otimes \ydiagram{1,1} = \ydiagram{2,2} \oplus \ydiagram{1,1,1,1} = \ydiagram{2,2} \oplus \ydiagram{1}, \qquad \bar{\mathbf{3}} \otimes \bar{\mathbf{3}} = (0,2) \oplus (1,0) = \bar{\mathbf{6}} \oplus \mathbf{3}.$$

Combining the diquark-antidiquark products give four cross terms

$$\ydiagram{2} \otimes \ydiagram{2,2} = \ydiagram{4,2} \oplus \ydiagram{3,2,1} \oplus \ydiagram{2,2,2}, \qquad \mathbf{6} \otimes \bar{\mathbf{6}} = (2,2) \oplus (1,1) \oplus (0,0) = \mathbf{27} \oplus \mathbf{8} \oplus \mathbf{1},$$

$$\ydiagram{2} \otimes \ydiagram{1} = \ydiagram{3} \oplus \ydiagram{2,1}, \qquad \mathbf{6} \otimes \mathbf{3} = (3,0) \oplus (1,1) = \mathbf{10} \oplus \mathbf{8},$$

$$\ydiagram{1,1} \otimes \ydiagram{2,2} = \ydiagram{3,3} \oplus \ydiagram{3,2,1}, \qquad \bar{\mathbf{3}} \otimes \bar{\mathbf{6}} = (0,3) \oplus (1,1) = \overline{\mathbf{10}} \oplus \mathbf{8},$$

$$\ydiagram{1,1} \otimes \ydiagram{1} = \ydiagram{2,1} \oplus \ydiagram{1,1,1}, \qquad \bar{\mathbf{3}} \otimes \mathbf{3} = (1,1) \oplus (0,0) = \mathbf{8} \oplus \mathbf{1},$$

The full decomposition gives

$$\boxed{\mathbf{3}^{\otimes 2} \otimes \bar{\mathbf{3}}^{\otimes 2} = \mathbf{27} \oplus \mathbf{10} \oplus \overline{\mathbf{10}} \oplus 4(\mathbf{8}) \oplus 2(\mathbf{1})} \tag{1}$$

which satisfies

$$27 + 10 + 10 + 4(8) + 2(1) = 81 = 3^4.$$

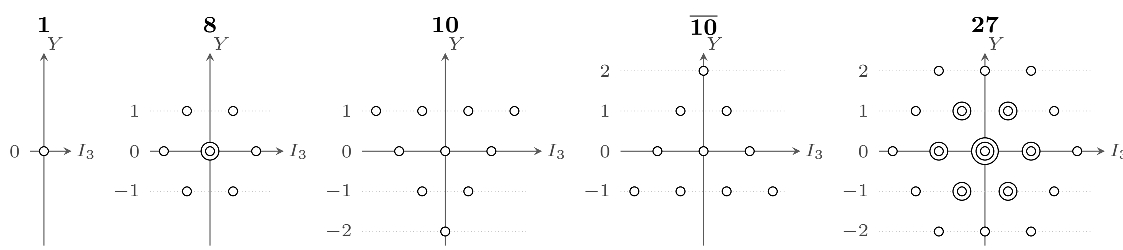


Figure 1: Flavour weight diagrams for the irreducible representations appearing in the light-tetraquark decomposition, plotted in the (I_3, Y) plane. Concentric circles indicate the multiplicity of coincident weights (two rings = multiplicity two, three rings = multiplicity three).

Pentaquark multiplets: $\mathbf{3}^{\otimes 4} \otimes \bar{\mathbf{3}}$

A pentaquark carries the content $qqqq\bar{q}$. The four-quark representation is constructed first, one quark at a time, and the antiquark is attached afterwards.

The four-quark representation

Starting from $\mathbf{3} \otimes \mathbf{3} = [2] \oplus [1,1]$ and multiplying by a further fundamental,

$$\ydiagram{2} \otimes \ydiagram{1} = \ydiagram{3} \oplus \ydiagram{2,1}, \qquad \ydiagram{1,1} \otimes \ydiagram{1} = \ydiagram{2,1} \oplus \ydiagram{1,1,1},$$

so that

$$\mathbf{3}^{\otimes 3} = [3] \oplus 2\,[2,1] \oplus [1,1,1] = \mathbf{10} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1}.$$

Attaching the fourth quark to each diagram in turn gives

$$\ydiagram{3} \otimes \ydiagram{1} = \ydiagram{4} \oplus \ydiagram{3,1},$$

$$\ydiagram{2,1} \otimes \ydiagram{1} = \ydiagram{3,1} \oplus \ydiagram{2,2} \oplus \ydiagram{2,1,1},$$

$$\ydiagram{1,1,1} \otimes \ydiagram{1} = \ydiagram{2,1,1},$$

where the last line contains a single term because placing the fourth box beneath the existing column would generate a column of four, which is forbidden in SU(3).

Since $[2,1]$ occurs twice in $\mathbf{3}^{\otimes 3}$, the second line enters with multiplicity two. The accumulated multiplicities are

$$[4] : 1, \qquad [3,1] : 1+2 = 3, \qquad [2,2] : 2, \qquad [2,1,1] : 2+1 = 3,$$

so that

$$\mathbf{3}^{\otimes 4} = [4] \oplus 3\,[3,1] \oplus 2\,[2,2] \oplus 3\,[2,1,1].$$

Converting row lengths to Dynkin labels through Eq. (**??**),

$$[4] \leftrightarrow (4,0), \qquad [3,1] \leftrightarrow (2,1), \qquad [2,2] \leftrightarrow (0,2), \qquad [2,1,1] \leftrightarrow (1,0),$$

and therefore

$$\mathbf{3}^{\otimes 4} = (4,0) \oplus 3(2,1) \oplus 2(0,2) \oplus 3(1,0) = \mathbf{15'} \oplus 3(\mathbf{15}) \oplus 2(\bar{\mathbf{6}}) \oplus 3(\mathbf{3}), \tag{2}$$

with $15+3(15)+2(6)+3(3) = 81 = 3^4$. Here $\mathbf{15'} \equiv (4,0)$ denotes the totally symmetric fifteen-plet and $\mathbf{15} \equiv (2,1)$ the mixed-symmetry one.

Attaching the antiquark

Multiplying each term of Eq. (2) by $\bar{\mathbf{3}} = [1,1]$,

$$\ydiagram{4} \otimes \ydiagram{1,1} = \ydiagram{5,1} \oplus \ydiagram{4,1,1}, \qquad (4,0) \otimes (0,1) = (4,1) \oplus (3,0),$$

$$\ydiagram{3,1} \otimes \ydiagram{1,1} = \ydiagram{4,2} \oplus \ydiagram{4,1,1} \oplus \ydiagram{3,2,1}, \qquad (2,1) \otimes (0,1) = (2,2) \oplus (3,0) \oplus (1,1),$$

$$\ydiagram{2,2} \otimes \ydiagram{1,1} = \ydiagram{3,3} \oplus \ydiagram{3,2,1}, \qquad (0,2) \otimes (0,1) = (0,3) \oplus (1,1),$$

$$\ydiagram{2,1,1} \otimes \ydiagram{1,1} = \ydiagram{3,2,1} \oplus \ydiagram{2,2,2}, \qquad (1,0) \otimes (0,1) = (1,1) \oplus (0,0),$$

with dimension checks $45 = 35 + 10$, $45 = 27 + 10 + 8$, $18 = 10 + 8$ and $9 = 8 + 1$.

After each line is weighted by the multiplicity with which its parent appears in Eq. (2), the terms sum to

$$\boxed{\mathbf{3}^{\otimes 4}\otimes\bar{\mathbf{3}}=\mathbf{35}\oplus 4(\mathbf{10})\oplus 3(\mathbf{27})\oplus 2(\overline{\mathbf{10}})\oplus 8(\mathbf{8})\oplus 3(\mathbf{1})} \tag{3}$$

where $(4,1)=\mathbf{35}$, $(3,0)=\mathbf{10}$, $(2,2)=\mathbf{27}$, $(0,3)=\overline{\mathbf{10}}$, $(1,1)=\mathbf{8}$ and $(0,0)=\mathbf{1}$. The dimensions balance,

$$35+4(10)+3(27)+2(10)+8(8)+3(1)=35+40+81+20+64+3=243=3^5,$$

and the three singlets match the multiplicity of $\mathbf{3}$ in Eq. (2).

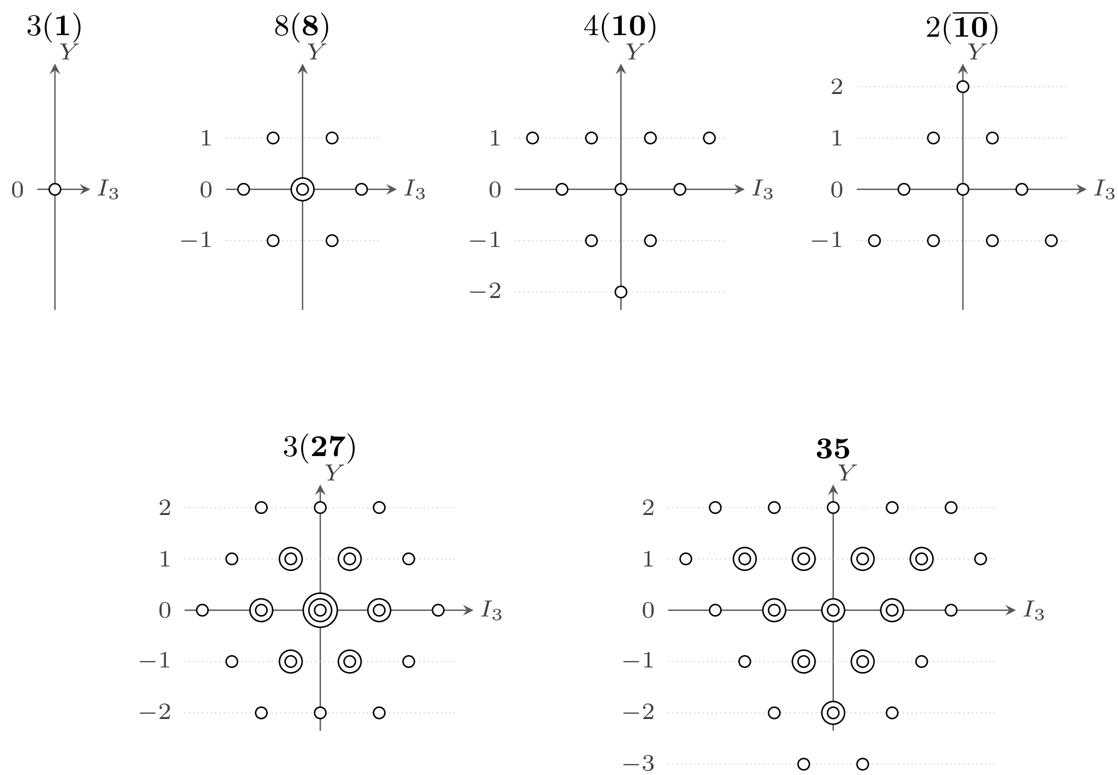


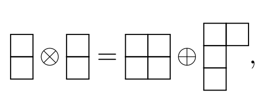
Figure 2: Flavour weight diagrams for the distinct irreducible representations appearing in the light-pentaquark decomposition $\mathbf{3}^{\otimes 4}\otimes\bar{\mathbf{3}}$. Each distinct weight diagram is plotted only once. Concentric circles indicate the multiplicity of coincident weights within one copy of the corresponding irreducible representation.

Antipentaquark multiplets: $\bar{\mathbf{3}}^{\otimes 4}\otimes\mathbf{3}$

An antipentaquark carries the flavour content $\bar{q}\bar{q}\bar{q}\bar{q}q$. The four-antiquark representation is constructed first, one antiquark at a time, and the remaining quark is attached afterwards. The resulting decomposition is the complex conjugate of the pentaquark decomposition in Eq. (3).

The four-antiquark representation

An antiquark transforms in the anti-fundamental representation $\bar{\mathbf{3}}=[1,1]$. Combining two antiquarks gives



and therefore

$$\bar{\mathbf{3}} \otimes \bar{\mathbf{3}} = (0,2) \oplus (1,0) = \bar{\mathbf{6}} \oplus \mathbf{3}.$$

The tableau $\ydiagram{2,1,1}$ contains a complete column of three boxes. Removing this column leaves $\ydiagram{1}$, so the two tableaux represent the same SU(3) multiplet.

Attaching a third antiquark gives

$$\ydiagram{2,2} \otimes \ydiagram{1,1} = \ydiagram{3,3} \oplus \ydiagram{3,2,1},$$

$$\ydiagram{2,1,1} \otimes \ydiagram{1,1} = \ydiagram{3,2,1} \oplus \ydiagram{2,2,2}.$$

Thus,

$$\bar{\mathbf{3}}^{\otimes 3} = [3,3] \oplus 2[3,2,1] \oplus [2,2,2] = \overline{\mathbf{10}} \oplus 2(\mathbf{8}) \oplus \mathbf{1}.$$

Attaching the fourth antiquark to each diagram gives

$$\ydiagram{3,3} \otimes \ydiagram{1,1} = \ydiagram{4,4} \oplus \ydiagram{4,3,1},$$

$$\ydiagram{3,2,1} \otimes \ydiagram{1,1} = \ydiagram{4,3,1} \oplus \ydiagram{4,2,2} \oplus \ydiagram{3,3,2},$$

$$\ydiagram{2,2,2} \otimes \ydiagram{1,1} = \ydiagram{3,3,2}.$$

Terms containing more than three rows have been omitted because they are not allowed as SU(3) Young diagrams.

Since $[3,2,1]$ occurs twice in $\bar{\mathbf{3}}^{\otimes 3}$, the second product enters with multiplicity two. The accumulated multiplicities are

$$[4,4] : 1, \qquad [4,3,1] : 1+2=3, \qquad [4,2,2] : 2, \qquad [3,3,2] : 2+1=3.$$

Therefore,

$$\bar{\mathbf{3}}^{\otimes 4} = [4,4] \oplus 3[4,3,1] \oplus 2[4,2,2] \oplus 3[3,3,2].$$

For a tableau with row lengths $[r_1, r_2, r_3]$, the corresponding Dynkin labels are

$$(p,q) = (r_1 - r_2, r_2 - r_3).$$

Consequently,

$$[4,4] \leftrightarrow (0,4), \qquad [4,3,1] \leftrightarrow (1,2),$$
$$[4,2,2] \leftrightarrow (2,0), \qquad [3,3,2] \leftrightarrow (0,1).$$

The four-antiquark decomposition is therefore

$$\bar{\mathbf{3}}^{\otimes 4} = (0,4) \oplus 3(1,2) \oplus 2(2,0) \oplus 3(0,1) = \overline{\mathbf{15}'} \oplus 3(\overline{\mathbf{15}}) \oplus 2(\mathbf{6}) \oplus 3(\bar{\mathbf{3}}). \tag{4}$$

The dimensions satisfy

$$15 + 3(15) + 2(6) + 3(3) = 81 = 3^4.$$

Attaching the quark

The remaining quark transforms in the fundamental representation $\mathbf{3} = [1]$. Multiplying each term of Eq. (4) by $\mathbf{3}$ gives

$$\ydiagram{4,4} \otimes \ydiagram{1} = \ydiagram{5,4} \oplus \ydiagram{4,4,1}, \qquad (0,4)\otimes(1,0) = (1,4)\oplus(0,3),$$

$$\ydiagram{4,3,1} \otimes \ydiagram{1} = \ydiagram{5,3,1} \oplus \ydiagram{4,4,1} \oplus \ydiagram{4,3,2}, \qquad (1,2)\otimes(1,0) = (2,2)\oplus(0,3)\oplus(1,1),$$

$$\ydiagram{4,2,2} \otimes \ydiagram{1} = \ydiagram{5,2,2} \oplus \ydiagram{4,3,2}, \qquad (2,0)\otimes(1,0) = (3,0)\oplus(1,1),$$

$$\ydiagram{3,3,2} \otimes \ydiagram{1} = \ydiagram{4,3,2} \oplus \ydiagram{3,3,3}, \qquad (0,1)\otimes(1,0) = (1,1)\oplus(0,0).$$

The corresponding dimension checks are

$$45 = 35 + 10, \qquad 45 = 27 + 10 + 8, \qquad 18 = 10 + 8, \qquad 9 = 8 + 1.$$

After weighting each product by the multiplicity of its parent representation in Eq. (4), the contributions are

$$\begin{gathered}
\overline{\mathbf{15}'} \otimes \mathbf{3} = \overline{\mathbf{35}} \oplus \overline{\mathbf{10}}, \\
3\big(\overline{\mathbf{15}} \otimes \mathbf{3}\big) = 3(\mathbf{27}) \oplus 3(\overline{\mathbf{10}}) \oplus 3(\mathbf{8}), \\
2\big(\mathbf{6} \otimes \mathbf{3}\big) = 2(\mathbf{10}) \oplus 2(\mathbf{8}), \\
3\big(\bar{\mathbf{3}} \otimes \mathbf{3}\big) = 3(\mathbf{8}) \oplus 3(\mathbf{1}).
\end{gathered}$$

Combining identical representations gives

$$\boxed{\bar{\mathbf{3}}^{\otimes 4} \otimes \mathbf{3} = \overline{\mathbf{35}} \oplus 4(\overline{\mathbf{10}}) \oplus 3(\mathbf{27}) \oplus 2(\mathbf{10}) \oplus 8(\mathbf{8}) \oplus 3(\mathbf{1})} \tag{5}$$

where

$$\begin{gathered}
(1,4) = \overline{\mathbf{35}}, \qquad (0,3) = \overline{\mathbf{10}}, \qquad (2,2) = \mathbf{27}, \\
(3,0) = \mathbf{10}, \qquad (1,1) = \mathbf{8}, \qquad (0,0) = \mathbf{1}.
\end{gathered}$$

The dimensions balance,

$$35 + 4(10) + 3(27) + 2(10) + 8(8) + 3(1) = 35 + 40 + 81 + 20 + 64 + 3 = 243 = 3^5.$$

The antipentaquark result is the complex conjugate of the pentaquark decomposition. In particular,

$$\mathbf{35} \longleftrightarrow \overline{\mathbf{35}}, \qquad \mathbf{10} \longleftrightarrow \overline{\mathbf{10}},$$

while $\mathbf{27}$, $\mathbf{8}$, and $\mathbf{1}$ are self-conjugate.

Gell-Mann–Okubo mass relations

For an octet gives the mass relation

$$2\left(M_{Y=1} + M_{Y=-1}\right) = 3M_{Y=0,I=0} + M_{Y=0,I=1}.$$

Thus, if three of the four distinct mass groups in an octet are known, the mass of the fourth can be predicted.

For a decuplet, the GMO formula reduces to an equal-spacing rule,

$$M_{Y=0} - M_{Y=1} = M_{Y=-1} - M_{Y=0} = M_{Y=-2} - M_{Y=-1}.$$

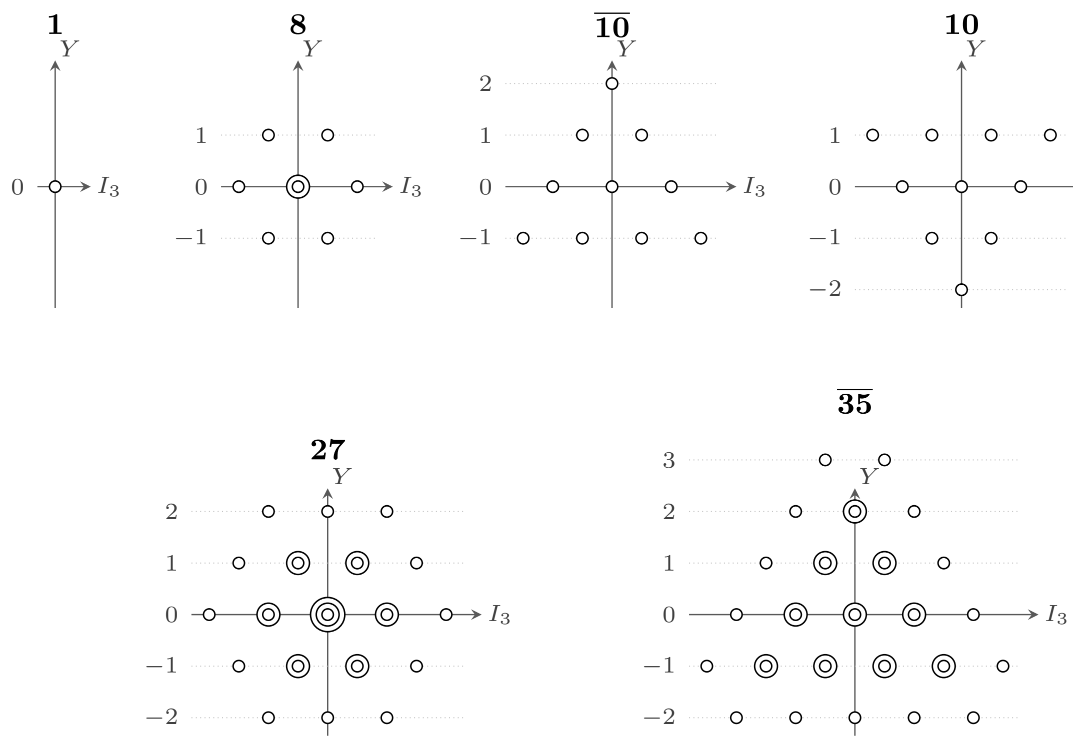


Figure 3: Flavour weight diagrams for the distinct irreducible representations appearing in the light anti-pentaquark decomposition $\overline{\mathbf{3}}^{\otimes 4} \otimes \mathbf{3}$. The complete decomposition is $\overline{\mathbf{35}} \oplus 3(\mathbf{27}) \oplus 4(\overline{\mathbf{10}}) \oplus 2(\mathbf{10}) \oplus 8(\mathbf{8}) \oplus 3(\mathbf{1})$. Each distinct weight diagram is displayed once. Concentric circles indicate the multiplicity of coincident weights within a single copy of the corresponding irreducible representation.

Similarly, for an antidecuplet,

$$M_{Y=1} - M_{Y=2} = M_{Y=0} - M_{Y=1} = M_{Y=-1} - M_{Y=0}.$$

The 27-plet and 35-plet derived above can also be analyzed using the GMO formula, but for simple relations doesn't exist. Each (Y, I) value can be substituted directly into the general GMO equation. Once three independent masses in a proposed multiplet are known, M_0, α, and β can be fitted, allowing the masses of the remaining states to be predicted.

Application to observed tetraquark candidates. Most experimentally observed exotic hadrons contain charm quarks. Since the charm quark is not part of light-flavor $SU(3)$, a hidden-charm pair $c\bar{c}$ behaves as a singlet under $SU(3)_{\text{flavor}}$. A hidden-charm tetraquark therefore has the light-flavor decomposition

$$c\bar{c}q\bar{q}: \qquad 3 \otimes \bar{3} = 8 \oplus 1.$$

This makes experimentally observed hidden-charm tetraquarks useful for testing the octet GMO relation. Two particularly interesting states are $Z_c(3900)$ and $Z_{cs}(4000)$ are assumed to belong to the same flavor octet, the octet GMO relation becomes

Table 1: Selected hidden-charm tetraquark candidates relevant to a possible flavor-octet assignment.

State	Flavor	Y	I	Mass (MeV)
$Z_c(3900)$	non-strange	0	1	3887.1 ± 2.6
$Z_{cs}(4000)$	strange	+1	$\frac{1}{2}$	3988 ± 5
$\overline{Z}_{cs}(4000)$	anti-strange	−1	$\frac{1}{2}$	3988 ± 5
X_8	isoscalar octet partner	0	0	unknown

$$2\left(M_{Z_{cs}} + M_{\overline{Z}_{cs}}\right) = 3M_{X_8} + M_{Z_c}.$$

Since Z_{cs} and its antiparticle have same mass, so

$$4M_{Z_{cs}} = 3M_{X_8} + M_{Z_c}.$$

The mass of the missing isoscalar member is therefore

$$M_{X_8} = \frac{4M_{Z_{cs}} - M_{Z_c}}{3}.$$

Using the measured masses in Table 1 gives approximately

$$M_{X_8} = \frac{4(3988) - 3887.1}{3} \simeq 4022 \text{ MeV}.$$

Comparison with observed pentaquarks. For hidden-charm pentaquarks of the form $c\bar{c}qqq$, the $c\bar{c}$ pair is a singlet under $SU(3)$, so the three light quarks transform as

$$3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1.$$

Several non-strange and strange hidden-charm pentaquark candidates have been reported by LHCb, as summarized in Table 2 [4, 1, 3].

A possible pentaquark octet would contain states with

$$(Y, I) = (1, \tfrac{1}{2}),\ (0, 0),\ (0, 1),\ (-1, \tfrac{1}{2}),$$

and would satisfy

Table 2: Selected experimentally observed hidden-charm pentaquark candidates. The states are not assumed to belong to the same flavor multiplet.

State	Strangeness S	Y	I	Mass (MeV)
$P_c(4312)^+$	0	1	$\frac{1}{2}$	4311.9
$P_c(4440)^+$	0	1	$\frac{1}{2}$	4440.3
$P_c(4457)^+$	0	1	$\frac{1}{2}$	4457.3
$P_{cs}(4338)^0$	-1	0	0	4338.2
$P_{cs}(4459)^0$	-1	0	0	4458.8

$$2\left(M_{Y=1} + M_{Y=-1}\right) = 3M_{Y=0,I=0} + M_{Y=0,I=1}.$$

Unlike the tetraquark case, this equation contains two unknown masses. Since pentaquarks have baryon number $B = 1$, the $Y = -1$ state is a doubly-strange pentaquark rather than the antiparticle of the $Y = 1$ state. Therefore, one additional octet member must be discovered before GMO can predict the fourth.

Conclusion

To finalize, $SU(3)$ flavor symmetry and Gell-Mann-Okubo mass relations can be used to classify tetraquarks and pentaquarks and relate them to the masses of each other. This can be done by first using Young Tableuax to determine the possible representations in $SU(3)$, then graphing the isospin to hypercharge to show possible states. Once this is done, we can combine them through their groups to relate masses of each one in their respective groups.

The reason this is important is because using these relations we can predict the masses and properties of particles that have not yet been discovered. For example, the CERN Large Hadron Collider can detect the structure of certain exotic hadrons by colliding protons at extremely high speeds. However, much of tetraquarks and pentaquarks have not been discovered. Using the methods in this paper, we have successfully predicted what these undiscovered hadrons look like along with their masses. When these particles do get discovered, we now have a baseline for what they will act like.

References

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